Solving Stochastic Climate-Economy Models Using Least-Squares Monte Carlo Method

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Integrated assessment models for climate change

- Since the early 1990s, various Integrated Assessment Models (IAMs) for climate change have emerged as a mainstream analytical tool in climate change economics
  - cost-benefit models: **DICE**, **FUND**, **PAGE**
  - process-based models: **IMAGE**, **MESSAGE-GLOBIOM**, **AIM/CGE**, **GCAM**, **REMIND-MAgPIE** and **WITCH-GLOBIOM**, etc. `www.iamconsortium.org/resources/models-documentation/`
- Key results of IPCC Assessment Reports on mitigation and climate impacts have primarily relied on the process-based IAMs from various research groups
- IPCC – the Intergovernmental Panel on Climate Change, the United Nations body for assessing the science related to climate change `www.ipcc.ch/`.
- Climate change research community has developed Shared Socioeconomic Pathways (SSPs): SSP1 – SSP5 and Representative Concentration Pathways (RCPs).

Talk outline

- Dynamic integrated climate-economy (**DICE**) model
- Stochastic DICE (**SDICE**) models
- Regional Integrated model of Climate and the Economy (**RICE**)
- Least Squares Monte Carlo (**LSMC**) method: regression on sampled state/control variables to estimate conditional expectation in dynamic programming solving optimal stochastic control problem:

  Longstaff and Schwartz (2001) and Tsitsiklis and Van Roy (2001) for valuation of American options. The convergence properties of this method are examined in Belomestny et al. (2010); Belomestny (2011), and Aïd, Langrené and Pham (2014). Extended with control randomisation in Kharroubi, Langrené, and Pham (2014) to handle endogenous state variables; adjustments for expected utility problems in Andréasson and Shevchenko (2022).

  LSMC variants: regress now versus regress later, realised value versus regression surface

DICE model

- The DICE model introduced by Prof William Nordhaus (Yale University) in 1992 is an extremely popular (even though widely criticized!) integrated assessment model (IAM) for the joint modeling of economic and climate systems.

- Prof Nordhaus was the winner of the Nobel Prize in Economic Sciences in 2018 for integrating climate change into a long-run macroeconomic analysis.

- DICE has been regularly revised over the last three decades with the first version dating back to technical report Nordhaus et al. (1992) , and the most recent revision being DICE-2023/2016 Nordhaus (2017).
- The DICE model is a deterministic approach that combines a Ramsey-Cass-Koopmans neoclassical model of economic growth (also known as the Ramsey growth model) with a simple climate model.
- The DICE model is one of the three main IAMs (the other two are FUND and PAGE) used by the United States government to determine the social cost of carbon; see [7]. It balances parsimony with realism and is well documented with all published model equations; in addition, its code is publicly available, which is an exception rather than the rule for IAMs.

$$V_0(\boldsymbol{X}_0) = \sup_{\boldsymbol{\mu},\boldsymbol{c}} \left[\sum_{t=0}^{\infty} e^{-\widetilde{\rho}\Delta t} U(c_t, L_t) \right]$$

where $U(c_t, L_t) = \frac{\Delta L_t}{1-\alpha} \left((c_t/L_t)^{1-\alpha} - 1 \right)$ is utility of consumption, and state vector $\boldsymbol{X}_t = (K_t, \boldsymbol{M}_t, \boldsymbol{T}_t)$ is evolving over time as

$$K_{t+1} = K_t(1-\delta_K)^{\Delta} + \Delta \times (Q_t(K_t, T_t, \mu_t) - c_t)$$
$$\boldsymbol{M}_{t+1} = \boldsymbol{\Phi}^M \boldsymbol{M}_t + \Delta \times (\widetilde{\beta} E_t(K_t, \mu_t), 0, 0)'$$
$$\boldsymbol{T}_{t+1} = \boldsymbol{\Phi}^T \boldsymbol{T}_t + \Delta \times \left(\xi_1 F_{t+1}(M_{t+1}^{AT}), 0 \right)'$$

- $\boldsymbol{c} = (c_0, c_1, \ldots)$ is the consumption $c_t > 0$ and $\boldsymbol{\mu} = (\mu_0, \mu_1, \ldots)$ is the carbon emission mitigation rate $\mu_t \geq 0$.
- $K_t > 0$ is the **world economic capital**, $\boldsymbol{M}_t = (M_t^{AT}, M_t^{UP}, M_t^{LO})'$ are the **carbon concentrations in the atmosphere, upper and lower oceans**. $\boldsymbol{T}_t = (T_t^{AT}, T_t^{LO})'$ are the **temperatures in atmosphere and lower oceans** measured in $^\circ\mathrm{C}$ above temperature in 1900.

$$\mathbf{\Phi}^M = \begin{pmatrix} \phi_{11} & \phi_{12} & 0 \\ \phi_{21} & \phi_{22} & \phi_{23} \\ 0 & \phi_{32} & \phi_{33} \end{pmatrix}, \ \mathbf{\Phi}^T = \begin{pmatrix} 1 - \frac{\xi_1 \eta}{\texttt{t2xco2}} - \xi_1\xi_3 & \xi_1\xi_3 \\ \xi_4 & 1-\xi_4 \end{pmatrix}$$

$$\beta = 1/3.666,\ \phi_{21} = 0.12,\ \phi_{32} = 0.007$$
$$\phi_{11} = 1 - \phi_{21},\ \phi_{33} = 1 - \phi_{23},\ \phi_{22} = 1 - \phi_{12} - \phi_{32}$$
$$\phi_{12} = \phi_{21} \times \frac{\texttt{mateq}}{\texttt{mueq}},\ \phi_{23} = \phi_{32} \times \frac{\texttt{mueq}}{\texttt{mleq}}$$

`mueq`=360 is equilibrium concentration in upper strata (GtC)
`mleq`=1720 is equilibrium concentration in lower strata (GtC)
`mateq`=588 is equilibrium concentration atmosphere (GtC)
`t2xco2`=3.1 is equilibrium temperature impact ($^\circ$C per doubling CO2)
$\eta = 3.6813$ is forcings of equilibrium CO2 doubling
$\xi_1 = 0.1005, \xi_3 = 0.088, \xi_4 = 0.025$ are temperature equation coefficients

- Net output: $Q_t(K_t, T_t, \mu_t) = \Omega_t(\mu_t, \sigma_t, T_t^{AT}) Y(A_t, K_t, L_t)$
- Total emission: $E_t(K_t, \mu_t) = (1 - \mu_t)\sigma_t Y(A_t, K_t, L_t) + E_t^{Land}$
- Radiative forcing: $F_t(M_t^{AT}) = \eta \log_2(M_t^{AT} / \widetilde{M}^{AT}) + F_t^{EX}$

Here:

$Y(A_t, K_t, L_t) = A_t K_t^{\gamma} L_t^{1-\gamma}$: **gross GDP** (Cobb-Douglas production function)

$\Omega_t(\mu_t, \sigma_t, T_t^{AT}) = 1 - \dfrac{\sigma_t 550(1 - 0.025)^t \mu_t^{\theta_2}}{1000\theta_2} - \pi_2 (T_t^{AT})^2$: **abatement-damage factor**

$A_t = A_{t-1}(1 + g_A(t-1))$: **total productivity factor**

$\sigma_t = \sigma_{t-1}(1 + g_\sigma(t-1))$: **decarbonisation function**

$L_t = L_{t-1} \, (11.500/L_{t-1})^{0.134}$: **world population** in billions

F_t^{EX} : **exogenous radiative forcings**

E_t^{Land} : **land emission**

- Deterministic DICE is solved using Excel Solver or GAMS by brute force solving optimisation problem wrt $\boldsymbol{c} = (c_0, \ldots, c_N)$ and $\boldsymbol{\mu} = (\mu_0, \ldots, \mu_N)$ simultaneously

$$(\boldsymbol{\mu}^*, \boldsymbol{c}^*) = \arg\sup_{\boldsymbol{\mu}, \boldsymbol{c}} \left[\sum_{t=0}^{N} e^{-\widetilde{\rho}\Delta t} U(c_t, L_t) \right],$$

In DICE2016: $N = 100$, $\Delta = 5$ years.

GAMS (General Algebraic Modeling System) is a high-level programming language for mathematical modeling https://www.gams.com/.

- The **social cost of carbon** (USD per ton) SCC is the monetized economic loss caused by a 1-metric-ton increase in atmospheric CO_2.

$$SCC_t = -1000\widetilde{\beta} \frac{\partial V_t / \partial M_t^{\mathrm{AT}}}{\partial V_t / \partial K_t}$$

- $A(t)$, $\sigma(t)$ and L_t are exogenous functions that can be calibrated to baseline SSPs (see our paper arxiv: 2504.11721)

The stochastic DICE model

Stochastic DICE can be formulated by adding stochastic shocks to the deterministic DICE model in various places. Then we solve

$$V_0(\boldsymbol{X}_0) = \sup_{\boldsymbol{\mu},\boldsymbol{c}} \mathbb{E}\left[\sum_{t=0}^{\infty} e^{-\widetilde{\rho}\Delta t} U(c_t, L_t)\right],$$

subject to the state vector $\boldsymbol{X}_t = (K_t, \boldsymbol{M}_t, \boldsymbol{T}_t, ...)$ stochastically evolving over time, ie we solve optimal stochastic control problem (i.e. optimal controls at time t are functions of $\boldsymbol{X}_t$).
For example, one can consider adding shocks to state variables

$$\begin{aligned}
K_{t+1} &= K_t(1-\delta_K)^{\Delta} + \Delta \times (Q_t(K_t, T_t, \mu_t) - c_t)e^{\epsilon^K_{t+1}},\\
\boldsymbol{M}_{t+1} &= \boldsymbol{\Phi}^M \boldsymbol{M}_t + \Delta \times (\widetilde{\beta} E_t(K_t, \mu_t), 0, 0)' e^{\epsilon^M_{t+1}},\\
\boldsymbol{T}_{t+1} &= \boldsymbol{\Phi}^T \boldsymbol{T}_t + \Delta \times \begin{pmatrix} \xi_1 F_{t+1}(M^{AT}_{t+1}) \\ 0 \end{pmatrix} + \boldsymbol{\epsilon}^T_{t+1},\ t = 0, 1, \ldots.
\end{aligned}$$

where $(\epsilon^K_t, \boldsymbol{\epsilon}^M_t, \boldsymbol{\epsilon}^T_t)$ are random disturbances.

- One can also add discrete shocks

$$I_{t+1} = \mathcal{T}^D(I_t, \epsilon_{t+1}^I)$$

e.g. affecting the gross output as

$$Y(A_t, K_t, L_t) = (1 - \chi(I_t))A_t K_t^\gamma L_t^{1-\gamma}$$

where $\chi(I_t)$ is representing impact of shocks; and $I_t \in \{0, 1\}$ is representing normal and stressed regimes.

- One can add shocks to e.g. the growth rates of the output and decarbonisation leading to new state variables

$$A_{t+1} = A_t(1 + \tilde{g}_t^A),\ \tilde{g}_t^A \sim N(g_t^A, \sigma_A^2)$$
$$\sigma_{t+1} = \sigma_t(1 + \tilde{g}_t^\sigma),\ \tilde{g}_t^\sigma \sim N(g_t^\sigma, \sigma_\sigma^2)$$

- One can consider parameter uncertainty (via adding state variables) e.g. in the damage function coefficient $\hat{\pi}_2 \sim N(\pi_2, \sigma_\pi^2)$, equilibrium temperature sensitivity $\mathtt{ETS} \sim LN(t2xco2, \sigma_{ETS}^2)$, etc.
- One can also consider Bayesian learning for parameters (via adding state variables for prior parameters).

SDICE can be solved using the dynamic programming

$$V_t(\boldsymbol{X}_t) = \sup_{\mu_t, c_t} \left(U(c_t, L_t) + e^{-\widetilde{\rho}\Delta} \mathbb{E}[V_{t+1}(\boldsymbol{X}_{t+1}) | \boldsymbol{X}_t] \right), \text{ s.t. } V_N(\boldsymbol{X}_N) = 0,$$

and the optimal strategy can be found as

$$(\mu_t^*(\boldsymbol{X}_t), c_t^*(\boldsymbol{X}_t)) = \arg\sup_{\mu_t, c_t} \left(U(c_t, L_t) + e^{-\widetilde{\rho}\Delta} \mathbb{E}[V_{t+1}(\boldsymbol{X}_{t+1}) | \boldsymbol{X}_t] \right).$$

Studies solving SDICE as a recursive dynamic programming using deterministic grid methods include:

- Kelly and Kolstad (1999) and Leach (2007) formulated the DICE model with stochasticity in the temperature-time evolution.
- Cai and Lontzek (2019) formulates DICE as a dynamic programming problem with a stochastic shock on the economy and climate.
- Traeger (2014) developed a DICE model with a smaller number of state variables.
- Lontzek, Cai, Judd, and Lentonet (2015) studied the impact of tipping points.
- Shevchenko et al. (2022) considered discrete shocks to the economy.

In the case of many state variables and controls, simulations methods such as the Least Squares Monte Carlo simulation method are needed.

- Numerically, stochastic DICE can be solved using deterministic grid in state variable space and evaluating value function between grid points via interpolation.
- There are few studies/attempts to solve stochastic DICE as an optimal stochastic control problem (finding decision under uncertainty) due to mathematical sophistical and numerical complexity and long computing time.
- In the case of one or two stochastic state variables, one can use quadrature methods to calculate expectation $\mathbb{E}[V_{t+1}(\boldsymbol{X}_{t+1})|\boldsymbol{X}_t]$. E.g. for an arbitrary function $f(x)$, the Gauss-Hermite quadrature is

$$\int_{-\infty}^{+\infty} e^{-z^2} f(z)dz \approx \sum_{j=1}^{q} W_j^{(q)} f(z_j^{(q)})$$

- In the case of many state variables and controls, simulations methods such as the Least Squares Monte Carlo simulation method are needed (Arandjelović, Shevchenko, Murakami, et al 2024).

Least-Squares Monte Carlo method for model extensions

The stochastic DICE model can be studied allowing multiple shocks.

- Additional states and stochasticities make a determinstic grid-quadrature based numerical solution computationally infeasible.
- Least-Squares Monte Carlo (LSMC) is an approximate method for solving stochastic control problems, e.g. Longstaff and Schwartz (2001) for valuation of American options.
- Essentially a simulation and regression algorithm, where random paths are simulated and the conditional expectation in Bellman equation is approximated with a regression function, then solved via backwards recursion for stochastic control problems.
- Original exogenous LSMC extended in Kharroubi et al. (2014) with endogenous state variables and control randomisation.
- Adjustments are needed to LSMC in the case of utility models, see Andréasson and Shevchenko (2022).

Problem Definition

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$ be a filtered complete probability space and $\mathcal{F}_t$ represents the information available up to time t. All the processes introduced below are adapted to $\{\mathcal{F}_t\}_{t\geq 0}$.

Notation:

- Controlled state variable $X^\pi = (X_t^\pi)_{t=t_0,\dots,T}$
- Control $\pi = (\pi_t)_{t=t_0,\dots,T}$
- Random disturbance $Z = (Z_t)_{t=t_0,\dots,T}$
- State variable evolution $X_{t+1}^\pi = T(X_t^\pi, \pi_t, Z_{t+1})$

Objective: maximise the expected value of the total reward

$$V_{t_0}(x) = \sup_{\pi} \mathbb{E}\left[\beta^{T-t_0} G_T(X_T^\pi) + \sum_{t=t_0}^{T-1} \beta^{t-t_0} R_t(X_t^\pi, \pi_t) | X_{t_0}^\pi = x\right],$$

where G_T and R_t are functions satisfying integrability conditions and β is discounting factor.

Problem Definition

This type of problem can be solved with backward recursion of the Bellman equation, where

$$V_T(x) = G_T(x),$$
$$V_t(x) = \sup_{\pi_t} \left\{ R_t(x, \pi_t) + \mathbb{E}\left[\beta V_{t+1}(X^{\pi}_{t+1}) | X^{\pi}_t = x; \pi_t \right] \right\}.$$

Optimal value of control is found as

$$\pi^*_t(x) = \arg \sup_{\pi_t \in \mathcal{A}_t} \left\{ R_t(x, \pi_t) + \mathbb{E}\left[\beta V_{t+1}(X^{\pi}_{t+1}) \;\middle|\; X^{\pi}_t = x; \pi_t \right] \right\}.$$

The solution of such problem is often not possible to find analytically and numerical methods are required.

As the number of state variables, stochastic processes, or control variables increases, the numerical solution becomes very expensive computationally and simulation methods such as LSMC are favoured.

- If the state variable is not affected by the control, the idea behind utilitising the LSMC method is to approximate the conditional expectation

$$\Phi_t(X_t) = \mathbb{E}\left[\beta V_{t+1}(X_{t+1})|X_t\right],$$

by a regression scheme with independent variables X_t, and response variable $\beta V_{t+1}(X_{t+1})$. The approximation of the function is then denoted as $\widehat{\Phi}_t$.

- If the state variable is affected by control, then techniques such as control randomization are required where the conditional expectation

$$\Phi_t(X_t^\pi, \pi_t) = \mathbb{E}\left[\beta V_{t+1}(X_{t+1}^\pi)|X_t^\pi; \pi_t\right]$$

is estimated by regression of $\beta V_{t+1}(X_{t+1})$ on X_t^π and randomised π_t [9]. Neural network regression works better in high-dimensional case.

- For ease of notation, the superscript π on the state variable is now dropped.

Arguments for LSMC

Arguments for LSMC:

- Does not suffer from "curse of dimensionality", hence faster than other numerical methods as the number of state variables increase.
- No restrictions on dynamics of stochastic processes (contrary to PDE's). Enough to be able to simulate a path.
- Parametric estimate in feedback form of control (no grid required).

Arguments against LSMC:

- Approximate method only, and can have substantial errors piling up over multiple periods.
- Can be computationally intensive, especially for the optimisation of control variables.
- Basis function can be difficult to find and is highly problem specific.

Let $\mathbf{L}(X_t, \pi_t)$ be a vector of basis functions and $\mathbf{\Lambda}_t$ corresponding regression coefficients such that

$$\mathbb{E}\left[\beta V_{t+1}(X_{t+1}) | X_t; \pi_t\right] = \mathbf{\Lambda}_t' \mathbf{L}(X_t, \pi_t).$$

If M independent Markovian paths of state and control variables are simulated, one can consider the ordinary linear regression

$$\beta V_{t+1}(X_{t+1}^m) = \mathbf{\Lambda}_t' \mathbf{L}(X_t^m, \pi_t^m) + \epsilon_t^m,$$
$$\epsilon_t^m \overset{iid}{\sim} F_t(\cdot), \;\; \mathbb{E}[\epsilon_t^m] = 0, \;\; \mathrm{var}[\epsilon_t^m] = \sigma_t^2, \;\; m = 1, ..., M$$

$$\hat{\mathbf{\Lambda}}_t = \arg\min_{\mathbf{\Lambda}} \sum_m \left[\beta V_{t+1}(X_{t+1}^m) - \mathbf{\Lambda}' \mathbf{L}(X_t^m, \pi_t^m)\right]^2.$$

The above is the so-called regress now LSMC.

LSMC for models with utility functions

There are difficulties with LSMC in the case of utility type model (difficult to fit due to extreme curvature over the full sample). It is better to regress the *transformed* value function.

Define a transformation H^{-1}; $H^{-1}(H(x)) = x$. Let $\mathbf{L}(X_t, \pi_t)$ be a vector of basis functions and $\mathbf{\Lambda}_t$ corresponding regression coefficients such that

$$\mathbb{E}\left[H^{-1}(\beta V_{t+1}(X_{t+1}))|X_t; \pi_t\right] = \mathbf{\Lambda}_t'\mathbf{L}(X_t, \pi_t).$$

If M independent Markovian paths of state and control variables are simulated, one can consider the ordinary linear regression

$$H^{-1}(\beta V_{t+1}(X_{t+1}^m)) = \mathbf{\Lambda}_t'\mathbf{L}(X_t^m, \pi_t^m) + \epsilon_t^m,$$
$$\epsilon_t^m \overset{iid}{\sim} F_t(\cdot), \;\; \mathbb{E}[\epsilon_t^m] = 0, \;\; \mathrm{var}[\epsilon_t^m] = \sigma_t^2, \;\; m = 1, ..., M$$

$$\hat{\mathbf{\Lambda}}_t = \arg\min_{\mathbf{\Lambda}} \sum_m \left[H^{-1}(V(t, X_t^m)) - \mathbf{\Lambda}'\mathbf{L}(X_t^m, \pi_t^m)\right]^2.$$

Our objective is to estimate $\Phi_t(X_t, \pi_t) = \mathbb{E}\left[\beta V_{t+1}(X_{t+1}) | X_t; \pi_t\right]$:

$$H^B(\mathbf{\Lambda}_t' \mathbf{L}(X_t, \pi_t)) := \Phi_t(X_t, \pi_t) = \int H(\mathbf{\Lambda}_t' \mathbf{L}(X_t, \pi_t) + \epsilon_t) dF_t(\epsilon_t),$$

where $F_t(\epsilon_t)$ is the distribution of disturbance term ϵ_t. Obviously,

$$\widehat{H}^B(\widehat{\mathbf{\Lambda}}_t' \mathbf{L}(X_t, \pi_t)) = H(\widehat{\mathbf{\Lambda}}_t' \mathbf{L}(X_t, \pi_t))$$

will be neither unbiased nor consistent unless the transformation is linear. If a specific distribution is assumed for ϵ_t, then the integration in can be performed. Otherwise, the empirical distribution of residuals

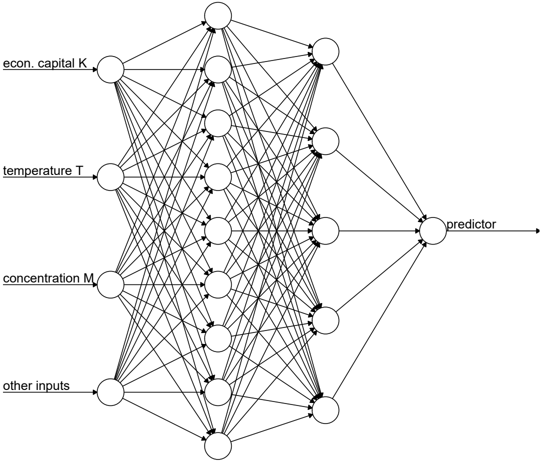
$$\widehat{\epsilon}_t^m = H^{-1}(\beta V_{t+1}(X_{t+1}^m)) - \widehat{\mathbf{\Lambda}}_t' \mathbf{L}(X_t^m, \pi_t^m),$$

can be used to perform the required integration leading to the following **Smearing Estimate**:

$$\widehat{H}^B(\widehat{\mathbf{\Lambda}}_t' \mathbf{L}(X_t, \pi_t)) = \frac{1}{M} \sum_{m=1}^{M} H(\widehat{\mathbf{\Lambda}}_t' \mathbf{L}(X_t, \pi_t) + \widehat{\epsilon}_t^m),$$

One can also use the Smearing Estimate with Controlled Heteroskedasticity (Andreasson and Shevchenko (2022))

Typical LSMC application assumes ordinary polynomials for $\mathbf{L}(X_t, \pi_t)$ but $\boldsymbol{\Lambda}_t'\mathbf{L}(X_t, \pi_t)$ can be replaced by Neural Network approximation $\mathcal{NN}_{\boldsymbol{\theta}_t}(X_t, \pi_t)$ that requires numerical optimisation to estimate NN parameters $\boldsymbol{\theta}_t$.



LSMC Algorithm: Endogenous state and random control

We utilise a discretised version of [10] with some modifications in forward simulation critical to get results for our problem: state variables are sampled at each time slice t with transition to $t+1$ instead of trajectories.

Algorithm Forward simulation

1: **for** $t = 0$ **to** $N-1$ **do**
2: **for** $m = 1$ **to** M **do**
[Simulate random samples]
3: $X_t^m := Rand \in \mathcal{X}$ ▷ State
4: $\widetilde{\pi}_t^m := Rand \in \mathcal{A}$ ▷ Control
5: $z_{t+1}^m := Rand \in \mathcal{Z}$ ▷ Disturbance
[Compute the state variable after control]
6: $\widetilde{X}_{t+1}^m := \mathcal{T}_t(X_t^m, \widetilde{\pi}_t^m, z_{t+1}^m)$ ▷ Evolution of state
7: **end for**
8: **end for**

Algorithm Backward solution (**Realised value - regress now**)

for $t = N$ **to** 0 **do**
  if $t = N$ **then** $\widehat{V}_t(\widetilde{\boldsymbol{X}}_t) := R_N(\widetilde{\boldsymbol{X}}_t)$
  else if $t < N$ **then**
    [Regression of transformed value function]
    $\widehat{\boldsymbol{\Lambda}}_t := \arg\min_{\boldsymbol{\Lambda}_t} \sum_{m=1}^{M} \left[\boldsymbol{\Lambda}_t' \mathbf{L}(X_t^m, \widetilde{\pi}_t) - H^{-1}(\beta \widehat{V}_{t+1}(\widetilde{X}_{t+1}^m))\right]^2$
    [Approximate conditional expectation] $\widehat{\Phi}_t(X_t, \widetilde{\pi}_t) := H^B(\widehat{\boldsymbol{\Lambda}}_t' \mathbf{L}(X_t, \widetilde{\pi}_t))$
    for $m = 1$ **to** M **do**
      $\widehat{X}_t^m := \widetilde{X}_t^m$
      [Optimal control] $\pi_t^*(\widehat{X}_t^m) := \arg\sup_{\pi_t \in A} \left\{ R_t(\widehat{X}_t^m, \pi_t) + \widehat{\Phi}_t(\widehat{X}_t^m, \pi_t) \right\}$
      [Update value function with optimal paths]
      $\widehat{V}_t(\widehat{X}_t^m) := R_t(\widehat{X}_t^m, \pi_t^*(\widehat{X}_t^m))$
      $\widehat{X}_{t+1}^m := \mathcal{T}_t(\widehat{X}_t^m, \pi_t^*(\widehat{X}_t^m), z_t^m)$
      for $t_j = t+1$ **to** $N-1$ **do**
        $\widehat{V}_t(\widehat{X}_t^m) := \widehat{V}_t(\widehat{X}_t^m) + \beta^{t_j - t} R_{t_j}(\widehat{X}_{t_j}^m, \pi_{t_j}^*(\widehat{X}_{t_j}^m))$
        $\widehat{X}_{t_j+1}^m := \mathcal{T}_t(\widehat{X}_{t_j}^m, \pi_{t_j}^*(\widehat{X}_{t_j}^m), z_{t_j}^m)$
      end for
      $\widehat{V}_t(\widehat{X}_t^m) := \widehat{V}_t(\widehat{X}_t^m) + \beta^{N-t} R_N(\widehat{X}_N^m)$
    end for
  end if
end for

Regression surface versus realised value

There are two alternative versions of the control randomisation algorithm [9]: the one that uses the regression surface to update the value function,

$$\widehat{V}_t(X_t) = R_t(X_t, \pi_t^*(X_t)) + \widehat{\Phi}_t(X_t, \pi_t^*(X_t)),$$

and another one that uses the realised value function,

$$\widehat{V}_t(X_t) = R_t(X_t, \pi_t^*(X_t)) + \beta \widehat{V}_{t+1}(X_{t+1}).$$

The first algorithm is the so-called value function iteration (VFI), while the second one is the so-called policy function iteration (PFI). The PFI requires recalculation of the sample paths for $t+1$ to T after each iteration backwards in time, as the optimal control affects the future state variables hence changes the simulated paths.

Algorithm Backward solution (**regression surface - regress later**)

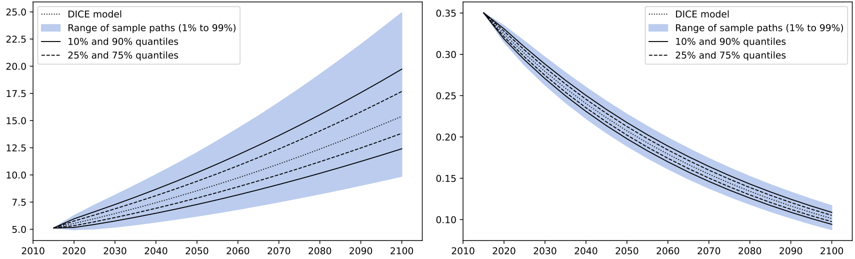
for $t = 0$ **to** N **do**
  for $m = 1$ **to** M **do**
    sample X_t^m in the domain of its possible values
  end for
end for

for $t = N$ **to** 0 **do**
  if $t = N$ **then**
    $\widehat{V}_t(X_t) := R_N(X_t)$
  else if $t < N$ **then**
    [Regression/approximation of value function]
    $\widehat{\mathbf{\Lambda}}_{t+1} := \arg\min_{\mathbf{\Lambda}_{t+1}} \sum_{m=1}^{M} \left[\mathbf{\Lambda}'_{t+1}\mathbf{L}(X_{t+1}^m) - \widehat{V}_{t+1}(X_{t+1}^m)\right]^2$
    Approximate value function $\widehat{V}_{t+1}(X_{t+1}) = \widehat{\mathbf{\Lambda}}'_{t+1}\mathbf{L}(X_{t+1})$
    for $m = 1$ **to** M **do**
      [Optimal control]
      $\pi_t^*(X_t^m) := \arg\sup_{\pi_t \in \mathcal{A}_t} \left\{R_t(X_t^m, \pi_t) + \beta \mathrm{E}[\widehat{V}_{t+1}(X_{t+1}) | X_t^m; \pi_t]\right\}$
      [Update value function]
      $\widehat{V}_t(X_t^m) := R_t(X_t^m, \pi_t^*) + \beta \mathrm{E}[\widehat{V}_{t+1}(X_{t+1}) | X_t^m; \pi_t^*]$
    end for
  end if
end for

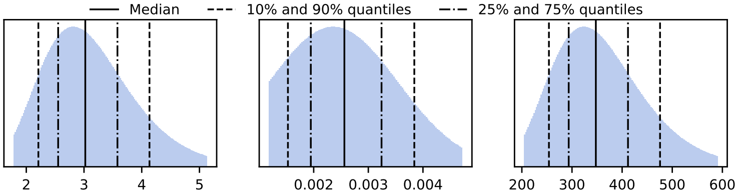
Stochastic DICE via LSMC

Most significant parameter uncertainties identified in Nordhaus (2018) are equilibrium temperature sensitivity parameter, the damage function coefficient, productivity growth rate, the rate of decarbonization and carbon-cycle coefficient. Thus, for LSMC illustration we consider

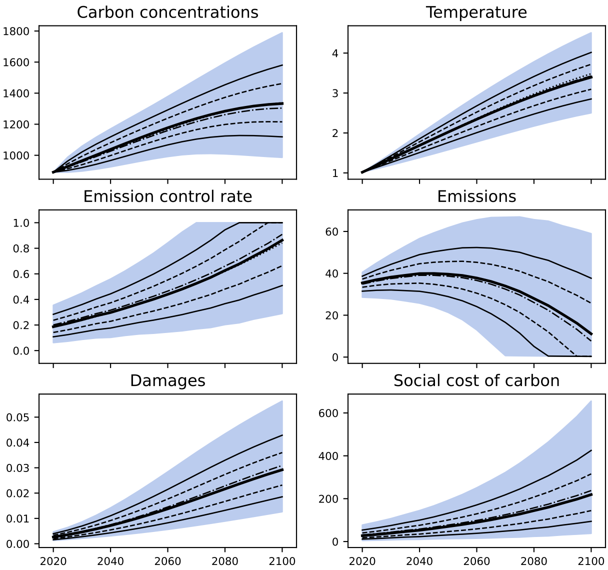
- **Productivity growth rate** $A_{t+1} = A_t/(1-\tilde{g}_t^A)$, $\tilde{g}_t^A \sim N(g_t^A, \sigma_A^2)$, where $g_A(t)$ is DICE2016 growth rate, $\sigma_A = 0.056e^{-0.005t\Delta}$
- **Decarbonisation rate** $\sigma_{t+1} = \sigma_t \exp(\tilde{g}_t^\sigma \Delta)$, $\tilde{g}_t^\sigma \sim N(g_t^\sigma, \sigma_\sigma^2)$, where g_t^σ is DICE2016 growth rate, $\sigma_\sigma = 0.0032(1-0.001)^{t\Delta}$
- **Equilibrium temperature sensitivity (ETS):** $\ln ETS \sim N(1.106, 0.2646^2)$. In deterministic DICE2016, it is `t2xco2`$=3.1$
- **Damage function coefficient**: $\pi_2 \sim N(0.00236, (0.00236/2)^2)$
- **Carbon-cycle coefficient**: $\ln CC \sim N(5.851, 0.2649^2)$. In deterministic DICE2016, it is equilibrium concentration in upper strata `mueq`$=360$ (GtC)

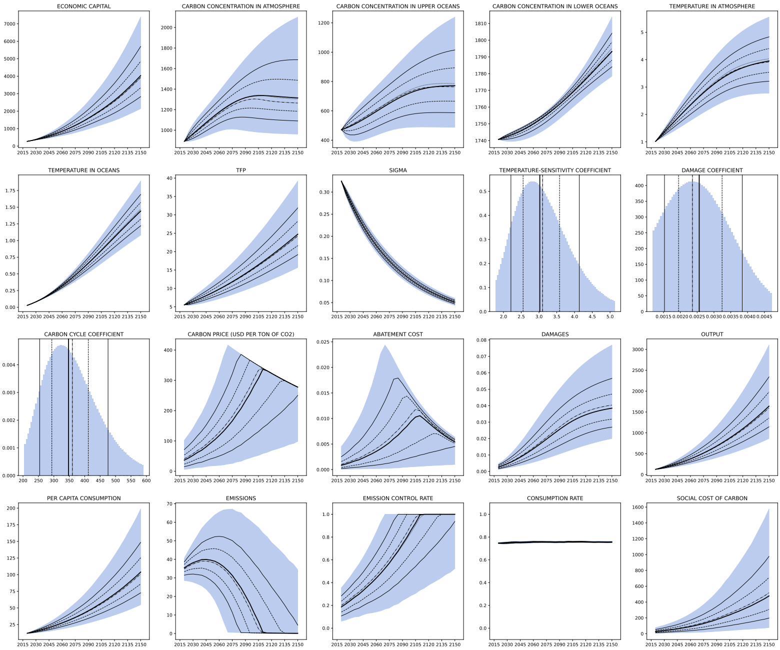


Total factor productivity A_t (left panel) and carbon intensity σ_t (right panel) under the assumption that the growth rates are uncertain.



Density plots of the parameter distributions of equilibrium temperature sensitivity (left panel), the damage coefficient (middle panel), and carbon cycle coefficient (right panel).


Carbon concentrations
1800
1600
1400
1200
1000
Temperature
4
3
2
1
Emission control rate
1.0
0.8
0.6
0.4
0.2
0.0
Emissions
60
40
20
0
Damages
0.05
0.04
0.03
0.02
0.01
0.00
2020
2040
2060
2080
2100
Social cost of carbon
600
400
200
0
2020
2040
2060
2080
2100



ECONOMIC CAPITAL
CARBON CONCENTRATION IN ATMOSPHERE
CARBON CONCENTRATION IN UPPER OCEANS
CARBON CONCENTRATION IN LOWER OCEANS
TEMPERATURE IN ATMOSPHERE
TEMPERATURE IN OCEANS
TFP
SIGMA
TEMPERATURE-SENSITIVITY COEFFICIENT
DAMAGE COEFFICIENT
CARBON CYCLE COEFFICIENT
CARBON PRICE (USD PER TON OF CO2)
ABATEMENT COST
DAMAGES
OUTPUT
PER CAPITA CONSUMPTION
EMISSIONS
EMISSION CONTROL RATE
CONSUMPTION RATE
SOCIAL COST OF CARBON

Statistics for major variables

Variable	Mean	BG	Median	SD	IQR	CV
Social cost of carbon, 2020	30.9	28.3	28.7	12.5	16.7	0.40
Temperature, 2100 ($^\circ$C)	3.42	3.49	3.40	0.46	0.64	0.13
Carbon concentration, 2100 (ppm)	1,342	1,344	1,339	156	217	0.12
World output, 2100 (trillions, 2015$)	833.6	795.9	811.2	203.6	271.9	0.24
Emissions, 2100	14.0	13.1	12.0	13.3	23.6	0.95
Damages, 2100 (percent output)	3.0	2.9	2.9	1.0	1.4	0.34

Notes: SD, IQR and CV refer to standard deviation, interquartile range and coefficient of variation, respectively. BG refers to best guess, which is the value calculated along the expected trajectory, assuming that uncertainties are set to their respective means.

First-order (left) and total-order (right) Sobol indices

First-order:

	TFP	SIG	TSC	DC	CC
SCC	0.0	0.0	36.7	55.1	2.5
TAT	1.4	0.4	62.3	12.7	20.9
MAT	1.6	0.9	31.2	36.2	26.1
OUT	100.0	0.0	0.0	0.0	0.0
EMI	3.9	0.4	40.0	43.5	1.5
DAM	0.6	0.2	36.5	45.6	13.7

Total-order:

	TFP	SIG	TSC	DC	CC
SCC	0.0	0.0	42.2	60.6	2.9
TAT	2.2	0.4	64.2	14.3	21.4
MAT	4.8	1.0	33.5	38.7	26.2
OUT	100.0	0.0	0.0	0.0	0.0
EMI	10.2	0.7	49.0	52.6	2.3
DAM	0.9	0.2	38.5	48.0	15.9

Let $Y = f(\boldsymbol{X})$, $\boldsymbol{X} = (X_1, ..., X_d)'$ and $V_i = \mathrm{Var}_{X_i}(\mathrm{E}_{\boldsymbol{X}_{-i}}(Y|X_i))$.

First order (main effect) sensitivity indices: $S_i = \frac{V_i}{\mathrm{Var}(Y)}$; these indices represent the contribution of a single input variable to the output variance, ignoring interaction effects with other variables.

Total effect sensitivity indices: $S_{Ti} = \frac{\mathrm{E}_{\boldsymbol{X}_{-i}}(\mathrm{Var}_{X_i}(Y|\boldsymbol{X}_{-i}))}{\mathrm{Var}(Y)}$; these indices represent the contribution of an input variable to the output variance, including all interactions with other variables.

Variance-based sensitivity analysis - Sobol's method

$$Y = f_0 + \sum_{i=1}^{d} f_i(X_i) + \sum_{i<j}^{d} f_{ij}(X_i, X_j) + ... + f_{1,2,...,d}(X_1, X_2, ..., X_d)$$

$$f_0 = E[Y]$$

$$f_i(X_i) = E(Y|X_i) - f_0$$

$$f_{ij}(X_i, X_j) = E(Y|X_i, X_j) - f_0 - f_i - f_j$$

$$...$$

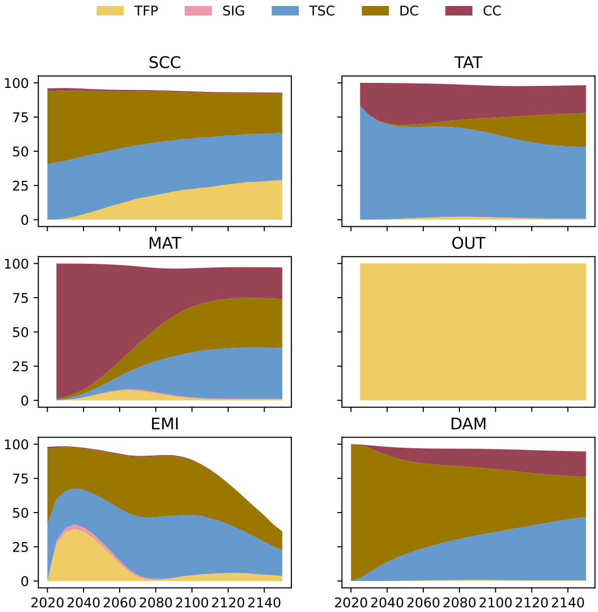
$$Var(Y) = \sum_{i=1}^{d} V_i + \sum_{i<j}^{d} V_{ij} + ... + V_{12...d}$$

$$V_i = \mathrm{Var}_{X_i}(\mathrm{E}_{\boldsymbol{X}_{-i}}(Y|X_i))$$

$$V_{ij} = \mathrm{Var}_{X_{ij}}(\mathrm{E}_{\boldsymbol{X}_{-ij}}(Y|X_i, X_j)) - V_i - V_j$$

$$...$$

Sobol's method (Sobol's indices) was introduced by Ilya Meyerovich Sobol' (15 Aug 1926 – 9 Dec 2025).



First-order Sobol' indices for main variables over time.

Numerical solution settings

Neural network:

- 11-dimensional input, 2 hidden layers with 20 nodes each and tanh() activation function;
- trained in Python / Keras using mini-batch stochastic gradient descent (Adam optimizer); early stopping is used; input covariates are standardized and output data are scaled to be in the range [-1, +1].

LSMC regress now/regression surface on a laptop with 64 cores (parallellization was used) took around 18 hours for 2^{23} samples, 9 hours for 2^{22} samples, and 4.5 hours for 2^{21} samples. Note that Cai and Lontzek (2019) study required the use of a few million core hours on a BlueWaters modern supercomputer with over 110,000 cores used. The final results are based on 10^5 forward trajectories.

For details, see our paper **A. Arandjelović, P.V. Shevchenko, T. Matsui, D. Murakami, T.A. Myrvoll (2024). Solving stochastic climate-economy models: A deep least-squares Monte Carlo approach.** `http://arxiv.org/abs/2408.9642`

RICE model

- Regional Integrated model of Climate and the Economy (RICE): Prof William Nordhaus and Prof Zili Yang in 1990, few updates with the latest in 2024.
- The general preference function is a social welfare function over $N = 12$ regions (US, EU, Japan, Russia, Eurasia, China, India, Middle East, Sub-Saharan Africa, Latin America, Other high income countries, and Other developing countries).
- Each region has individual consumption, population, capital stock, technology, emission control.
- Climate change is an externality phenomenon faced by all regions.
- Different nations can take different strategies for carbon reduction emission
  - **Market policies solution:** there are no controls on the emissions
  - **Cooperative policies solution:** carbon emission reduction is treated fully cooperatively in a globally efficient way
  - **Noncooperative solution:** nations undertake policies in their national self-interests and ignore their spillovers to other nations - **Nash equilibrium**

RICE model: cooperative case

$$V_0(\boldsymbol{X}_0) = \sup_{\boldsymbol{\mu},\boldsymbol{c}} \left[\sum_{n=1}^{N} \sum_{t=0}^{\infty} \varphi_{n,t} e^{-\widetilde{\rho}\Delta t} U(c_{n,t}, L_{n,t}) \right], \; \sum_{n} \varphi_{n,t} = N$$

subject to the state vector $\boldsymbol{X}_t = (\boldsymbol{K}_t, \boldsymbol{M}_t, \boldsymbol{T}_t)$ evolving over time as

$$K_{n,t+1} = K_{n,t}(1-\delta_K)^{\Delta} + \Delta \times (Q_{n,t}(K_{n,t}, T_t, \mu_{n,t}) - c_{n,t}), \; n = 1, ..., N$$

$$\boldsymbol{M}_{t+1} = \boldsymbol{\Phi}^M \boldsymbol{M}_t + \Delta \times (\widetilde{\beta} \sum_{n=1}^{N} E_{n,t}(K_{n,t}, \mu_{n,t}), 0, 0)',$$

$$\boldsymbol{T}_{t+1} = \boldsymbol{\Phi}^T \boldsymbol{T}_t + \Delta \times \begin{pmatrix} \xi_1 F_t(M_t^{AT}) \\ 0 \end{pmatrix}$$

$$Q_{n,t}(K_t, T_t, \mu_t) = \Omega_t(\mu_{n,t}, \sigma_{n,t}, T_{n,t}^{AT}) Y(A_{n,t}, K_{n,t}, L_{n,t}),$$
$$E_{n,t}(K_t, \mu_t) = (1 - \mu_{n,t})\sigma_{n,t} Y(A_{n,t}, K_{n,t}, L_{n,t}) + E_{n,t}^{Land},$$
$$F_t(M_t^{AT}) = \eta \log_2(M_t^{AT}/\widetilde{M}^{AT}) + F_t^{EX},$$

where

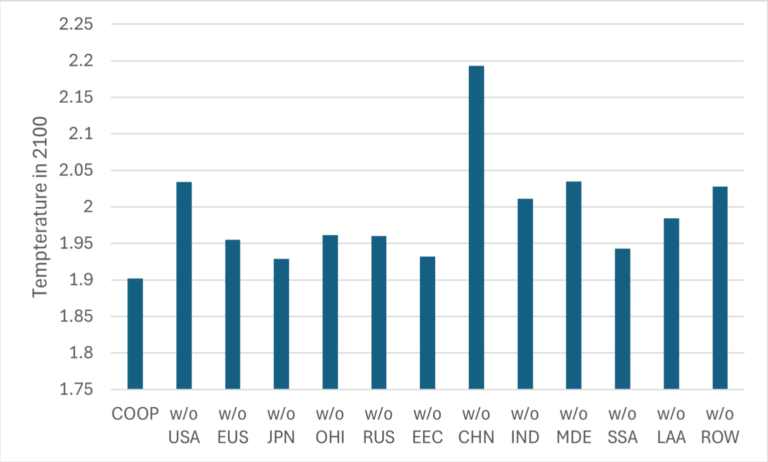
$$Y(A_{n,t}, K_{n,t}, L_{n,t}) = A_{n,t} K_{n,t}^{\gamma} L_{n,t}^{1-\gamma} \text{ (Cobb-Douglas production function)}$$
$$\Omega_{n,t}(\mu_{n,t}, \sigma_{n,t}, T_t^{AT}) = 1 - \theta_{n,1}\mu_{n,t}^{\theta_{n,2}} - \pi_{n,2}[T_t^{AT}]^{\delta_n^T},$$
$$A_{n,t} = A_{n,t-1}(1 + g_A(n, t-1)),\ \sigma_{n,t} = \sigma_{n,t-1} e^{g_{n,t-1}\Delta}$$

The **social cost of carbon** (USD per ton) is the monetized economic loss caused by a 1-metric-ton increase in atmospheric CO_2.

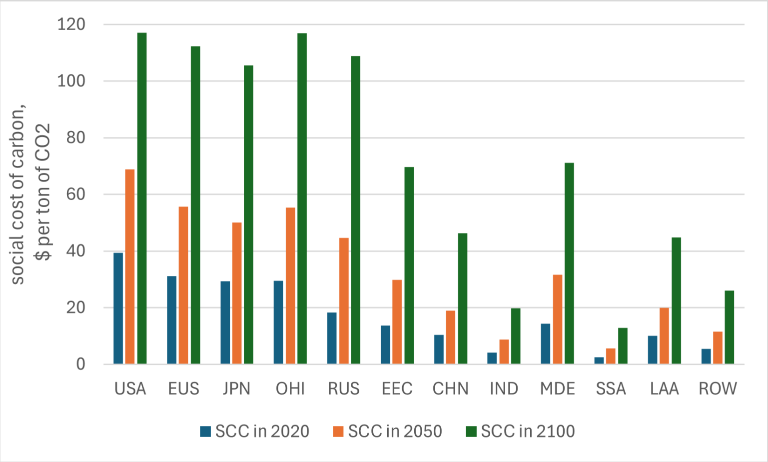
$$SCC_{n,t} = -1000\widetilde{\beta}\frac{\partial V_t/\partial M_t^{\mathrm{AT}}}{\partial V_t/\partial K_{n,t}}$$

RICE2020

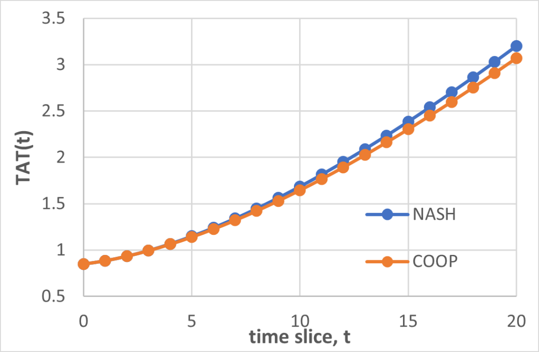
Cooperative solution under scenario of zero industrial emission by 2050 in all regions except one.



Social cost of carbon from cooperative solution under scenario of zero industrial emission by 2050 in all regions.



RICE2020



while emission control/social cost of carbon under NASH are about 10 times smaller compared to COOP.

Stochastic extensions: work in progress

Only the case of two regions is solved in the literature in the presence of stochastic shocks using deterministic grid method (Cai et al. 2018).

To model effects of the tipping events, we introduce state variable J_t: $J_t = 0$ corresponds to no tipping event triggered, $J_t = 1$ corresponds to triggered tipping point event. Then

$$Q_{i,t} = (1 - D_i(J_t))\Omega_{i,t} Y_{i,t}$$

where $D(J_t)$ is the impact of tipping point event, i.e. $D_i(0) = 0$ and $D(1) = d_i$. The matrix of transition probabilities $\Pr[J_{t+1}|J_t]$:

$$\begin{pmatrix} 1 - p_t & p_t \\ 0 & 1 \end{pmatrix}, \; p_t = 1 - \exp(-\varrho \max(0, T_t^{\mathrm{AT}} - T^*))$$

We follow Cail and Lontzek (2019) and Cai, et al (2018).

RICE2020 with tipping events

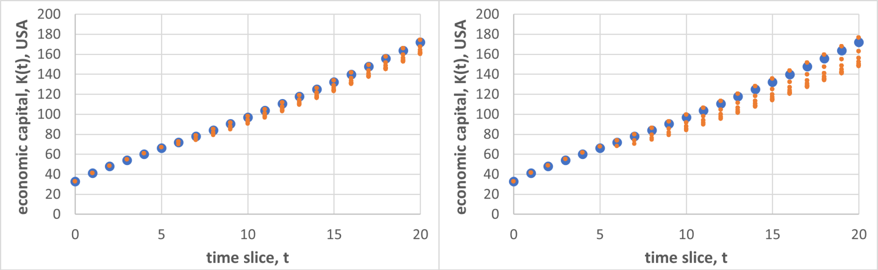


Figure: The same damage for all regions: 5% damage (left figure), 10% damage (right figure). 100,000 simulations, hazard rate 0.0035, $T^* = 1°C$.

So far we also considered shock to economic output

$$K_{i,t+1} = K_{i,t}(1-\delta_K)^{\Delta} + \Delta \times \left(Q_{i,t}\mathrm{e}^{\epsilon_{t+1}} - C_{i,t}\right)$$

where $\epsilon_t \sim N(-\sigma^2/2, \sigma^2)$ so that $E[\mathrm{e}^{\epsilon_t}] = 1$, and stochastic shocks in temperature

$$T_{t+1} = \Phi^T T_t + (\xi_1 F_t, 0)^\top + (\epsilon_{t+1}, 0)^\top, \epsilon_t \sim N(0, \sigma^2)$$

We used LSMC regress later (both for NASH and COOP cases), and regression with the 3rd order polynomials with respect to $K_{1,t}$,..., $K_{12,t}$, M_t^{AT}, M_t^{UP}, M_t^{LO}, T_t^{AT}, T_t^{OC}, i.e. $17 \times 3 = 51$ covariates. We checked that adding mixed terms or higher order terms does not improve the fit.

Final remarks

- LSMC can successfully be used for stochastic DICE and RICE models including recent versions of DICE/RICE.
- Lindahl weights equilibrium is problematic for LSMC but in principle can be handled via forward simulation approach with policies/value functions parameterised using NNs estimated by optimizing global objective function; e.g. in the vein of Friedl, Kübler, Scheidegger, Usui (2023) or Valatis and Villa (2024).
- It is possible to add emission permit trading between regions (Cai, Malik, Shin 2023), or/and interactions between regions (Cai, Brock, Xepapadeas, Judd 2018).

Thank you!

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